On general error distributions

R. Vasudeva[†], J. Vasantha Kumari

Department of Studies in Statistics, University of Mysore, Mysore-570006, India

Abstract. General error distributions are used in statistical modeling, where in the errors are not necessarily normally distributed. In this paper, we establish some interesting properties of these distributions and obtain a characterization.

1. Introduction

In statistical modeling, such as linear models, the difference between the observed value and the expected value is called an error. For many decades, it was assumed that the error random variable (r.v.) follows a normal distribution with mean zero. In many situations, it was observed that the normality is not an appropriate assumption. As alternatives, Subbotin (1923) introduced a class of distributions that are symmetric, but with variation in kurtosis. He noted that these distributions have many structural properties close to a normal distribution. This class of distributions is called the general error distributions. Nelson (1991) has developed linear regression models and time series models with heavy tails, assuming the underlying distribution as general error distribution (GED). Levy (2004) and Nadarajah (2005) discuss many distributional properties of a GED. In our paper, we call this class of distributions as general error distributions of the first kind, denoted by GED-I, as we will be discussing one more class of general error distributions (asymmetric).

The probability density function (pdf) of GED-I is given by

$$f_v(x) = \frac{v \exp\left\{-\frac{1}{2}\left|\frac{x}{\lambda}\right|^v\right\}}{\lambda 2^{1+1/v}\Gamma(\frac{1}{v})}, \quad v > 0, \ x \in \mathbb{R},$$

where $\lambda = \left[2^{-\frac{2}{v}}\frac{\Gamma(\frac{1}{v})}{\Gamma(\frac{3}{v})}\right]^{\frac{1}{2}}$ and $\Gamma(\cdot)$ denotes the Gamma function. It is well known that if X is GED-I, $EX = 0$, $EX^2 = 1$ and $EX^k < \infty$ for all $k > 0$. When $v = 2$, GED-I reduces to a standard normal distribution and when $v = 1$, it reduces to a double exponential distribution. Peng et al. (2009) established that the tail of the distribution function (d.f.) F of GED-I has the asymptotic relation, $1 - F(x) \sim x^{1-v}f(x)$, as $x \to \infty$, (where "$\sim$" means asymptotically equal). One can see that the tail of the d.f. is asymptotically Weibullian.

Another class of error distributions was introduced, by allowing the tail to be highly skewed (see, Wikipedia (2012)). We call this class as the general error distributions of the second kind, denoted as

GED-II. The pdf of GED-II is given by

$$g(x) = \frac{\exp\left\{-\frac{1}{2}\left(-\frac{1}{k}\log\left(1 - \frac{k(x-\xi)}{\alpha}\right)\right)^2\right\}}{\sqrt{2\pi}\alpha\left(1 - \frac{k(x-\xi)}{\alpha}\right)},$$

with $x \in \left(-\infty, \xi + \frac{\alpha}{k}\right)$ if $k > 0$, $x \in \left(\xi + \frac{\alpha}{k}, \infty\right)$ if $k < 0$, where ξ is a real constant and α a positive constant. When $k \to 0$, $g(\cdot)$ reduces to the pdf of a normal random variable (r.v.) with mean ξ and variance α^2. The d.f. G has the tail

$$1 - G(x) \sim \frac{\exp\left\{-\frac{1}{2}\left(-\frac{1}{k}\log\left(1 - \frac{k(x-\xi)}{\alpha}\right)\right)^2\right\}}{\sqrt{2\pi}\left(-\frac{1}{k}\log\left(1 - \frac{k(x-\xi)}{\alpha}\right)\right)},$$

as $x \to \infty$, when $k < 0$ and as $x \to \xi + \alpha/k$, when $k > 0$. The form of $1 - G(x)$ is well known when $k = 0$.

In this paper, we establish that the moment generating function (mgf) of a GED-I exists when $v \geq 1$ and fails to exist when $0 < v < 1$. We obtain a characterization and establish some additive properties of a GED-I. This is done in the next section. In Section 3, we show that the mgf of a GED-II fails to exist whenever $k \neq 0$. Also, we show that the tail $1 - G(x)$ is asymptotically, sandwiched between a Weibullian tail and a regularly varying tail. Throughout the paper, $M(t) = Ee^{tX}$, $-\infty < t < \infty$ denotes a mgf.

2. General error distribution I

Theorem 2.1. *Let a random variable X have GED-I with parameter $v > 0$. Then the mgf $M(t)$ exists for all t, when $v > 1$, exists in the region $(-\sqrt{2}, \sqrt{2})$ when $v = 1$, and fails to exist for any $t > 0$ when $0 < v < 1$.*

Proof. We have for any $v > 0$ that

$$M(t) = c_1 \int\limits_{-\infty}^{\infty} e^{tx} e^{-c_2|x|^v} dx, \ -\infty < t < \infty,$$

where $c_1 = v\left(\lambda 2^{1+1/v}\Gamma(\frac{1}{v})\right)^{-1}$ and $c_2 = (2\lambda^v)^{-1}$. Suppose that $0 < v < 1$. Let $t > 0$. Then for any $x > 0$

$$e^{tx}e^{-c_2|x|^v} = e^{tx\left(1 - \frac{c_2}{tx^{1-v}}\right)}.$$

Let $x_0 > 0$ be such that $\frac{c_2}{tx^{1-v}} < \frac{1}{2}$ for all $x \geq x_0$, so that

$$e^{tx\left(1 - \frac{c_2}{tx^{1-v}}\right)} \geq e^{\frac{tx}{2}}$$

for all $x \geq x_0$. Consequently,

$$\int\limits_{x_0}^{\infty} e^{tx} e^{-c_2|x|^v} dx = \infty,$$

which implies that $M(t)$ fails to exit (for any $t > 0$), whenever $0 < v < 1$.

When $v = 1$, the pdf $f(x) = \frac{1}{\sqrt{2}}e^{-\sqrt{2}|x|}$, $-\infty < x < \infty$. Here

$$M(t) = \frac{1}{\sqrt{2}} \int\limits_{-\infty}^{\infty} e^{tx - \sqrt{2}|x|} dx = \frac{1}{\sqrt{2}}(I_1 + I_2),$$

where

$$I_1 = \int\limits_{-\infty}^{0} e^{tx+\sqrt{2}x}dx \quad \text{and} \quad I_2 = \int\limits_{0}^{-\infty} e^{tx-\sqrt{2}x}dx.$$

Put $x = -y$. Then

$$I_1 = \int\limits_{0}^{\infty} e^{-y(t+\sqrt{2})}dy = \frac{1}{t+\sqrt{2}} \quad \text{and} \quad I_2 = \int\limits_{0}^{\infty} e^{tx-\sqrt{2}x}dx = \int\limits_{0}^{\infty} e^{-x(\sqrt{2}-t)}dx = \frac{1}{(\sqrt{2}-t)}.$$

Hence $M(t) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}+t} + \frac{1}{\sqrt{2}-t}\right) = 2((2-t^2))^{-1} = (1-\frac{t^2}{2})^{-1}$, whenever $|t| < \sqrt{2}$. Also, when $|t| \geq \sqrt{2}$, $M(t)$ fails to exit.

Let $v > 1$. Then

$$\begin{aligned} M(t) &= c_1 \int\limits_{-\infty}^{\infty} e^{tx-c_2|x|^v}dx \\ &= c_1 \left\{ \int\limits_{-\infty}^{0} e^{tx-c_2|x|^v}dx + \int\limits_{0}^{\infty} e^{tx-c_2 x^v}dx \right\} \\ &= c_1(I_1+I_2), \text{ say.} \end{aligned}$$

Since

$$tx - c_2 x^v = -c_2 x^v\left(1 - \frac{t}{c_2 x^{v-1}}\right) = -c_2 x^v(1+o(1)), \text{ as } x \to \infty,$$

one gets $I_2 < \infty$, for any real t.

Similarly, for $x < 0$ we have that

$$tx - c_2|x|^v = -c_2|x|^v\left(1 - \frac{tx}{c_2|x|^v}\right) = -c_2|x|^v\left(1 + \frac{t}{c_2|x|^{v-1}}\right) = -c_2|x|^v(1+o(1)), \text{ as } x \to -\infty.$$

Consequently, $I_1 < \infty$ for any t. In turn, $M(t)$ exists for all $t \in (-\infty, \infty)$. Applying Maclaurian expansion and identifying $M^k(0) = EX^k$, $k \geq 0$, one gets

$$M(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}EX^k.$$

Since X is symmetric about zero, we have $EX^k = 0$ for k odd. When k is even, $k = 2m$ say,

$$EX^{2m} = \left(\frac{\Gamma(\frac{1}{v})}{\Gamma(\frac{3}{v})}\right)^m \frac{\Gamma\left(\frac{2m+1}{v}\right)}{\Gamma(\frac{1}{v})}, \quad m \geq 1.$$

Hence

$$M(t) = \sum_{m=0}^{\infty} \frac{t^{2m}}{(2m)!}\left(\frac{\Gamma(\frac{1}{v})}{\Gamma(\frac{3}{v})}\right)^m \frac{\Gamma\left(\frac{2m+1}{v}\right)}{\Gamma(\frac{1}{v})}.$$

A closed form expression does not exist for $M(t)$. $\square$

Theorem 2.2. *A real valued random variable X, symmetric about 0, is GED-I if and only if for some $v > 0$, the random variable $\left(\frac{\Gamma(\frac{3}{v})}{\Gamma(\frac{1}{v})}X^2\right)^{v/2}$ has gamma distribution with parameter $1/v$.*

Proof. Let X be GED-I with parameter v. Put $Y = \left(\frac{\Gamma(\frac{3}{v})}{\Gamma(\frac{1}{v})}X^2\right)^{v/2}$. Then $|X| = \left(\frac{\Gamma(\frac{1}{v})}{\Gamma(\frac{3}{v})}\right)^{1/2} Y^{1/v}$. Let $Z = X^2$ and let the pdf of Z be $g(z)$. Then

$$g(z) = \frac{f(\sqrt{z})}{\sqrt{z}} = \frac{v}{\lambda 2^{1+1/v}\Gamma\frac{1}{v}} \, e^{-\frac{z^{\frac{v}{2}}}{2\lambda^v}} z^{\frac{1}{2}-1}, \, z > 0.$$

Denote by h, the pdf of Y. The relation, $Y = \left(\frac{\Gamma(\frac{3}{v})}{\Gamma(\frac{1}{v})}Z\right)^{\frac{v}{2}}$ gives $h(y) = g\left(\frac{y^{2/v}}{c}\right)\frac{2}{cv}y^{\frac{2}{v}-1}$, where $c = \frac{\Gamma(\frac{3}{v})}{\Gamma(\frac{1}{v})}$. Observe that $\lambda = 2^{-1/v}c^{-1/2}$. Substituting for $g(\cdot)$ and λ, on simplification, one gets

$$h(y) = \frac{1}{\Gamma(\frac{1}{v})}e^{-y}y^{\frac{1}{v}-1}, \, y > 0,$$

which is the pdf of a gamma r.v. with parameter $1/v$.

Conversely, let $Y = \left(\frac{\Gamma(\frac{3}{v})}{\Gamma(\frac{1}{v})}X^2\right)^{v/2}$ be Gamma$(1/v)$ and let H denote the d.f. of Y with pdf h. For any $y > 0$, defining $c = \Gamma(\frac{3}{v})/\Gamma(\frac{1}{v})$, we have

$$
\begin{aligned}
H(y) &= P(Y \le y) = P\left(\left(\frac{\Gamma(\frac{3}{v})}{\Gamma(\frac{1}{v})}X^2\right)^{\frac{v}{2}} \le y\right) \\
&= P\left(X \le \frac{y^{\frac{1}{v}}}{\sqrt{c}}\right) - P\left(X \le \frac{-y^{\frac{1}{v}}}{\sqrt{c}}\right). \quad (1)
\end{aligned}
$$

Differentiating (1) with respect to y and recalling that X is symmetric about 0, with pdf $f(\cdot)$, one gets

$$h(y) = 2f\left(\frac{y^{\frac{1}{v}}}{\sqrt{c}}\right)\frac{1}{\sqrt{c}v}y^{\frac{1}{v}-1}. \quad (2)$$

Put $x = y^{\frac{1}{v}}/\sqrt{c}$. Then (2) gives

$$
\begin{aligned}
f(x) &= \frac{\sqrt{c}\,v}{2}h\left(c^{\frac{v}{2}}x^v\right)(\sqrt{c}x)^{v-1} \\
&= \frac{\sqrt{c}\,v}{2}\frac{1}{\Gamma(\frac{1}{v})}e^{-c^{v/2}x^v}\left(c^{v/2}x^v\right)^{\frac{1}{v}-1}(\sqrt{c})^{v-1}x^{v-1} \\
&= \frac{\sqrt{c}\,v}{2\Gamma(\frac{1}{v})}e^{-c^{\frac{v}{2}}x^v}.
\end{aligned}
$$

Since $\lambda = 2^{-1/v}\left(\frac{\Gamma(\frac{1}{v})}{\Gamma(\frac{3}{v})}\right)^{\frac{1}{2}} = 2^{-1/v}c^{-1/2}$, for $x > 0$, one gets, $f(x) = v\left(\lambda 2^{1+\frac{1}{v}}\Gamma(\frac{1}{v})\right)^{-1}e^{-\frac{x}{2\lambda^v}}$. Also, $f(x) =$

$f(-x)$, $x > 0$, implies that $f(x) = v\left(\lambda 2^{1+\frac{1}{v}}\Gamma(\frac{1}{v})\right)^{-1}e^{-\frac{|x|^v}{2\lambda^v}}$, $-\infty < x < \infty$, which is the pdf of GED-I. $\square$

Theorem 2.3. *Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) GED-I random variables with parameter v. Then $\sum_{i=1}^{n}|X_i|^v$ is a gamma distributed random variable.*

Proof. Given that X is GED-I, proceeding as in the proof of Theorem 2.1, one can show that $|X|^v$ is Gamma$(1/(2\lambda^v), 1/v)$. By the closure property of gamma distribution, $X_1, X_2, \ldots, X_n$ are i.i.d. GED-I implies $\sum_{j=1}^{n}|X_j|^v$ is Gamma$(1/(2\lambda^v), n/v)$. $\square$

Remark 2.4. From the above theorem, we observe that if X_1 and X_2 are independent r.v.'s with a common GED-I, then $|X_1|^v$, $|X_2|^v$ and the convolution $|X_1|^v + |X_2|^v$ are gamma distributed random variables.

Remark 2.5. Convolution of GED-I, need not necessarily be GED-I. When $v = 1$, X_1 and X_2 are Laplace distributed random variables, but $X_1 + X_2$ is not Laplace distributed random variable. When $v = 2$, X_1 and X_2 are normal distributed random variables and $X_1 + X_2$ is also normal distributed random variable.

3. General error distributions II

Theorem 3.1. *The mgf $M(t)$ of a GED-II fails to exist for any $t > 0$ when $k < 0$ and for any $t < 0$ when $k > 0$.*

Proof. Consider the case $k < 0$. Then the pdf of GED-II is

$$g(x) = \frac{\exp\left\{-\frac{1}{2}\left(\frac{1}{(-k)}\log\left(1 + \frac{(-k)(x-\xi)}{\alpha}\right)\right)^2\right\}}{\sqrt{2\pi}\alpha\left(1 + \frac{(-k)(x-\xi)}{\alpha}\right)}, \quad x \geq \xi + \frac{\alpha}{k},$$

where $\alpha > 0$, $\xi \in (-\infty, \infty)$ are constants.

The mgf is given by $M(t) = \int_{(\xi+\frac{\alpha}{k})}^{\infty} e^{tx} g(x)dx$, $-\infty < t < \infty$. Put $1 + (-k)\frac{(x-\xi)}{\alpha} = y$. Then $x = \alpha(-k)^{-1}(y - 1) + \xi$ and hence

$$
\begin{aligned}
M(t) &= \frac{e^{(\xi+\frac{\alpha}{k})t}}{\sqrt{2\pi}\alpha} \int_0^{\infty} \left(e^{-\frac{\alpha y t}{k} - \frac{(\log y)^2}{2k^2} - \log y}\right)\left(-\frac{\alpha}{k}\right) dy \\
&= \frac{e^{(\xi+\frac{\alpha}{k})t}}{\sqrt{2\pi}(-k)} \int_0^{\infty} \left(e^{-\frac{\alpha y t}{k} - \frac{(\log y)^2}{2k^2} - \log y}\right) dy.
\end{aligned}
\tag{3}
$$

We have

$$-\frac{\alpha y t}{k} - \frac{(\log y)^2}{2k^2} - \log y = -\frac{\alpha y t}{k}\left(1 + \frac{k\left(\frac{(\log y)^2}{2k^2} + \log y\right)}{\alpha y t}\right).$$

Hence, for any given $t > 0$, one can find a $y_0 > 0$ such that $-\frac{\alpha y t}{k} - \frac{(\log y)^2}{k^2} - \log y \geq -\frac{\alpha y t}{2k}$. Consequently from (3),

$$M(t) \geq \frac{e^{(\xi+\frac{\alpha}{k})t}}{\sqrt{2\pi}(-k)} \int_{y_0}^{\infty} \left(e^{-\frac{\alpha y t}{2k}}\right) dy = \infty,$$

i.e. the mgf fails to exist (as it does not exist for any $t > 0$.)

Now consider the case, $k > 0$. Let X be GED-II with $k > 0$. Define $Y = -X$. The pdf of Y is

$$h(y) = \frac{\exp\left(-\frac{1}{2}\left(\frac{-1}{(-k')}\log\left(1 + \frac{(-k')(y-\xi')}{\alpha}\right)\right)^2\right)}{\sqrt{2\pi}\alpha\left(1 + \frac{(-k')(y-\xi')}{\alpha}\right)}, \quad y \geq \xi' + \frac{\alpha}{k'},$$

where $\xi' = -\xi$ and $k' = -k$. Hence Y is GED-II with $k' < 0$, consequently, Ee^{tY} fails to exist for any $t > 0$ and in turn $M(t) = Ee^{tX}$ fails to exist for any $t < 0$. $\square$

Theorem 3.2. *Let $\overline{H}_1(x)$, $x > 0$, denote the tail of a Weibullian d.f. and $H_2(x)$, $x > 0$, be a d.f. with regularly varying tail. Then for $k < 0$, the tail $\overline{G}$ of a GED-II satisfies the relation,*

$$\lim_{x \to \infty} \frac{\overline{H}_1(x)}{\overline{G}(x)} = 0 \quad and \quad \lim_{x \to \infty} \frac{\overline{H}_2(x)}{\overline{G}(x)} = \infty,$$

where $\overline{H}_2 = 1 - H_2$.

Proof. For $k < 0$, the tail of GED-II is given by

$$1 - G(x) \sim \frac{\exp\left\{-\frac{1}{2}\left(-\frac{1}{k}\log\left(1 + \frac{(-k)(x-\xi)}{\alpha}\right)\right)^2\right\}}{\sqrt{2\pi}\left(\frac{1}{(-k)}\log\left(1 + \frac{(-k)(x-\xi)}{\alpha}\right)\right)}. \tag{4}$$

Let $y = \left(1 + \frac{(-k)(x-\xi)}{\alpha}\right)$. Then the numerator on the right hand side expression of (4) is

$$\exp\left(-\frac{1}{2}\frac{1}{k^2}(\log y)(\log y)\right) = \exp\left(\log y^{-\frac{\log y}{2k^2}}\right) = y^{-\frac{\log y}{2k^2}}.$$

Observe that $y = \left(1 + \frac{(-k)(x-\xi)}{\alpha}\right) = \frac{(-k)x}{\alpha}\left(1 + \frac{(\frac{\alpha}{-k})-\xi}{x}\right)$ and hence $\log y = \log x + \log\frac{(-k)}{\alpha} + \log\left(1 + \frac{(\frac{\alpha}{-k})-\xi}{x}\right)$. Since $\left(1 + \frac{(\frac{\alpha}{-k})-\xi}{x}\right) \to 1$ as $x \to \infty$, given $\delta_1, \delta_2 > 0$, one can find a $x_0 > 0$ such that for all $x \geq x_0$,

$$\frac{(-k)}{\alpha}(1-\delta_1)x \leq y \leq \frac{(-k)}{\alpha}(1+\delta_1)x \quad and \quad \log x + \log(\frac{-k}{\alpha}) - \delta_2 \leq \log y \leq \log x + \log(\frac{-k}{\alpha}) + \delta_2.$$

Hence for all $x \geq x_0$,

$$\left(\frac{-k(1+\delta_1)x}{\alpha}\right)^{\frac{-\left(\log x + \log(\frac{-k}{\alpha}) + \delta_2\right)}{2k^2}} \leq y^{\frac{-\log y}{2k^2}} \leq \left(\frac{-k(1-\delta_1)x}{\alpha}\right)^{\frac{-\left(\log x + \log(\frac{-k}{\alpha}) - \delta_2\right)}{2k^2}}.$$

Also, the denominator on the right side of (4) satisfies for $x \geq x_0$,

$$\frac{\sqrt{2\pi}}{(-k)}\left(\log x + \log(\frac{-k}{\alpha}) - \delta_2\right) \leq \frac{\sqrt{2\pi}}{(-k)}\log y \leq \frac{\sqrt{2\pi}}{(-k)}\left(\log x + \log(\frac{-k}{\alpha}) + \delta_2\right).$$

Since $\log x + \log(\frac{-k}{\alpha}) \pm \delta_2 \sim \log x$, as $x \to \infty$, for a $\delta_3 > 0$, one can find a $x_1 \geq x_0$ such that for all $x \geq x_1$,

$$(1-\delta_3)\log x \leq \log x + \log(-\frac{k}{\alpha}) \pm \delta_2 \leq (1+\delta_3)\log x.$$

Put $d_1 = \frac{-k}{\alpha}(1-\delta_1)$ and $d_2 = \frac{-k}{\alpha}(1+\delta_1)$. Then for all $x \geq x_1$, the right hand side of (4) satisfies the inequality,

$$\frac{(-k)}{\sqrt{2\pi}}\frac{(d_2 x)^{\frac{-(1+\delta_3)\log x}{2k^2}}}{(1+\delta_3)\log x} \leq \frac{(-k)\exp\left(\frac{-1}{2k^2}\right)(\log y)^2}{\sqrt{2\pi}\log y} \leq \frac{(-k)}{\sqrt{2\pi}}\frac{(d_1 x)^{\frac{-(1-\delta_3)\log x}{2k^2}}}{(1-\delta_3)\log x}.$$

For a given $\delta_4 \in (0,1)$, one can hence find a $x_2 \geq x_1$ such that for all $x \geq x_2$,

$$\frac{(-k)}{\sqrt{2\pi}}\frac{(1-\delta_4)(d_2 x)^{\frac{-(1+\delta_3)\log x}{2k^2}}}{(1+\delta_3)\log x} \leq 1 - G(x) \leq \frac{(-k)}{\sqrt{2\pi}}\frac{(1+\delta_4)(d_1 x)^{\frac{-(1-\delta_3)\log x}{2k^2}}}{(1-\delta_3)\log x}.$$

Put $c_5 = \frac{(-k)}{\sqrt{2\pi}} \frac{(1-\delta_4)}{1+\delta_3}$ and $c_6 = \frac{(-k)}{\sqrt{2\pi}} \frac{(1+\delta_4)}{1-\delta_3}$. Then the above inequality can be written as

$$\frac{c_5(d_2 x)^{\frac{-(1+\delta_3)\log x}{2k^2}}}{\log x} \leq 1 - G(x) \leq \frac{c_6(d_1 x)^{\frac{-(1-\delta_3)\log x}{2k^2}}}{\log x}, \tag{5}$$

whenever $x \geq x_2$.

Let H_1 be a d.f. with Weibullian tail. Then $\overline{H}_1(x) = e^{-\alpha x^\beta} x^r (1+o(1))$, as $x \to \infty$, for some $\alpha > 0$, $\beta > 0$ and $r \in (-\infty, \infty)$. Also, let H_2 be a d.f. with regularly varying tail, i.e. $\overline{H}_2(x) = x^{-r} L(x)$, for some $r > 0$ and $L(\cdot)$ a slowly varying function. From (5) one can show that $\lim_{x \to \infty} \frac{\overline{H}_1(x)}{\overline{G}(x)} = 0$ and $\lim_{x \to \infty} \frac{\overline{H}_2(x)}{\overline{G}(x)} = \infty$, which completes the proof. $\square$

Remark 3.3. We have noticed that the tail thickness of GED-II is in between Weibullian and regularly varying tails. Perhaps, one reason for considering such a tail is that skewed distributions with Weibullian tail can be easily constructed from GED-I by truncating to the left (right) and skewed distributions with regularly varying tail will not have all moments finite, which is supposed to be an underlying structure of a random variable with GED.

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